determinancies, which give rise to three external or boundary compatibility conditions (CC_{22}, CC_{23}, and CC_{24}). The CC_{22} establishes deformation conformity between boundary nodes (1-5), likewise CC_{23} and CC_{24} constrains deformations along nodes (1,6,11) and nodes (5,4,3,2,1,6, and 11), respectively.

Discussion

Bandwidth of Compatibility Conditions

The upper bound to the bandwidth of compatibility conditions for a finite element model is ascertained from the location of elements in the discretization. For example, for the membrane shown in Fig. 1, the deformations of element 1, has to be compatible with its neighboring elements such as elements (1,2,3,9,10,11,12), which have been referred to as the lower cluster. The cluster consists of seven elements; each element has three force degrees of freedoms. The force degrees of freedom of the cluster is ($7 \times 3 = 21$). The maximum number of entries in any compatibility condition which belongs to this cluster is 21, which represents the upperbound bandwidth. The actual bandwidth of an individual compatibility condition is usually much smaller than its upperbound bandwidth. The lower cluster has eight compatibility conditions, (CC_1, $CC_2,\ldots,CC_7$ and CC_{19}). The upperbound bandwidth of 21 represents the composite bandwidth of all of the eight compatibility conditions of the cluster.

Upperbound Bandwidth of External Compatibility Conditions

The bandwidth of external compatibility conditions is examined taking the example of membrane boundary case I. Consider the elements (including their influencing elements) along the path connecting two or more boundary nodes. Let *EBW* represent the force degrees of freedom of all the elements along the path of the boundary nodes. The upperbound bandwidth of the external compatibility conditions is equal to *EBW*. Like the field compatibility conditions, the actual bandwidth of external compatibility condition is also much smaller than their upper bounds. The three external compatibility conditions (CC_{22},CC_{23},CC_{24}) have (4,2,19) entries respectively.

Computation Time

Computation time required to generate compatibility conditions for several structures are reported in Ref. 5. Here a rule of thumb is provided to estimate the upper and lower bounds to the computations required to generate the compatibility conditions. Consider a structure (n,m), and let the upper bound to the bandwidth of the $r=n-m$ compatibility conditions be represented by $(\gamma_1,\gamma_2,\ldots,\gamma_r)$ and actual bandwidths obtained be represented as $(\delta_1,\delta_2,\ldots,\delta_r)$. Let the total computational time required to solve r sets of linear equations of dimensions $(\gamma_1,\gamma_2,\ldots,\gamma_r)$ be represented by T_U. Likewise, time for the same when equation dimensions are changed to $(\delta_1,\delta_2,\ldots,\delta_r)$ be (T_L). The bounds to the computation time to generate the compatibility conditions of the structure are as follows:

$$\text{Upperbound time} = T_U \qquad \text{Lowerbound time} = T_L \tag{11}$$

Only the upperbound time T_U can be estimated before the generation of the compatibility conditions. The lowerbound time T_L, which represents a more realistic estimation, cannot be acertained a priori, since the actual bandwidths of compatibility conditions are known only after their generation. However, for a structure with many thousands of degrees of freedoms, the upperbound time T_U itself rerpresents a minor fraction of total computations required for the solution of the problem.

Conclusion

The generation of both field and boundary compatibility conditions from deformation displacement relations utilizing two key features: 1) compatibility bandwidth and 2) the node determinancy concept represents an efficient procedure to obtain the compatibility conditions of finite element models. The computer program generates sparse and banded compatibility conditions for a structure idealized by finite elements. The program requires no more additional input data than that is required by the finite element stiffness method. The bandwidth information and node determinancy condition are generated internally. We belive that the principle behind the generation of the compatibility conditions for finite element analysis, by and large, has been completed.

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Accurate Bending Analysis of Laminated Orthotropic Plates

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Introduction

MANY advances have been made in the theoretical development of higher-order theories for composite plates to account for shear deformation. A careful examination of the three-dimensional elasticity solution by Pagano[1] for a bidirectional composite plate reveals the nonlinearity of inplane displacements u and v in any layer with respect to thickness coordinate, as well as a sudden change of their slopes at the interfaces between two layers. By comparison with elasticity solution, Bert[2] evaluated different refined plate theories and concluded that a discrete layer version of a higher-order theory should be accurate.

Discrete layer theories proposed earlier incorporate either the slope discontinuity at an interface[3] or the nonlinear variation of u and v[4], but not both. Further they have the number of unknown variables increasing with the number of layers, thus making the analysis costly for practical laminates.

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A refined plate theory incorporating piecewise cubic variation of u and v through the thickness, slope discontinuity at the interfaces, and zero shear conditions on the lateral surfaces was proposed in Ref. 5. By enforcing the continuity of the transverse shear stresses at the interfaces, the number of unknown variables was kept independent of the number of layers. The theory was found to be very accurate for an infinite plate strip undergoing cylindrical bending. The theory presented herein is a modification of that of Ref. 5 and is valid for laminated orthotropic plates with finite boundaries, subjected to transverse loading in both the inplane directions.

Theory

The laminate composed of a finite number N of orthotropic layers is of constant thickness h. The plate is referred to as a Cartesian coordinate system x, y, and z such that it is simply supported on the boundaries $x = 0$, a and $y = 0$, b and is subjected to a transverse normal load $q\ (x,\ y)$ on the top surface $z = -h/2$. The axes of material symmetry are parallel to the plate axes x, y, and z. The deformations are assumed to be small and the material to be linearly elastic.

The displacement field is assumed to be of the form

$$u = u_0 - z(\partial w_0/\partial x) + p(z)u_1 + \sum_{i=1}^{N-1} [p(z) - p(z_i)]\phi_i(x,y)H(z - z_i)$$

$$v = v_0 - z(\partial w_0/\partial y) + p(z)v_1 + \sum_{i=1}^{N-1} [p(z) - p(z_i)]\psi_i(x,y)H(z - z_i)$$

$$w = w_0(x,y) \tag{1}$$

where $p(z) = z - (4z^3/3h^2)$.

In the aforementioned displacement field, $H(z)$ is the Heaviside unit step function, and the summation is extended over $N-1$ interfaces; z_i is the z coordinate of the ith interface, the one between ith and $(i+1)$th layers, and $u_0 = v_0$ for symmetric laminates.

By satisfying the transverse shear stress continuity in adjacent layers at each interface, the following expressions are obtained.

$$\phi_k = r_k u_1 \quad \text{and} \quad \psi_k = t_k v_1, \quad k = 1, 2, \ldots N-1$$

where

$$r_1 = \left(C_{55}^{(1)}/C_{55}^{(2)}\right) - 1, \qquad t_1 = \left(C_{44}^{(1)}/C_{44}^{(2)}\right) - 1$$

$$r_k = \left[\left(C_{55}^{(k)}/C_{55}^{(k+1)}\right) - 1\right]\left(1 + \sum_{s=1}^{k-1} r_s\right)$$

$$t_k = \left[\left(C_{44}^{(k)}/C_{44}^{(k+1)}\right) - 1\right]\left(1 + \sum_{s=1}^{k-1} t_s\right) \quad \text{for } k = 2,3,\ldots N-1 \tag{2}$$

The displacement variables are chosen in the form

$$w_0 = A_{mn} \sin\alpha x \sin\beta y$$

$$(u_0,u_1) = (B_{mn},D_{mn}) \cos\alpha x \sin\beta y$$

$$(v_0,v_1) = (C_{mn},E_{mn}) \sin\alpha x \cos\beta y \tag{3}$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$. These satisfy the boundary conditions: $w = w_{,xx} = N_x = 0$ at $x = 0$, a and $w = w_{,yy} = N_y = 0$ at $y = 0$, b.

When the terms involving the $H(z)$ are deleted in Eq. (1), the present model corresponds to the smeared laminate model of Reddy.[6]

The strains and stresses are expressed in terms of the coefficients A_{mn} to E_{mn} by using Eq. (3) in strain-displacement and plane-stress reduced constitutive relations. The total potential energy

$$U_t = \frac{1}{2}\int\int\int(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \int\int q w_0 \,\mathrm{d}x\,\mathrm{d}y$$

is obtained in terms of A_{mn} to E_{mn} as

$$\begin{aligned} U_t = {} & P_1A_{mn}^2 + P_2B_{mn}^2 + P_3C_{mn}^2 + P_4D_{mn}^2 \\ & + P_5E_{mn}^2 + P_6A_{mn}B_{mn} + P_7A_{mn}C_{mn} + P_8A_{mn}D_{mn} \\ & + P_9A_{mn}E_{mn} + P_{10}B_{mn}C_{mn} + P_{11}B_{mn}D_{mn} + P_{12}B_{mn}E_{mn} \\ & + P_{13}C_{mn}D_{mn} + P_{14}C_{mn}E_{mn} + P_{15}D_{mn}E_{mn} - P_{16}A_{mn} \end{aligned}$$

where P_i's (except P_{16}) are integrals over the thickness of the laminate and cannot be given in a general form for all laminations. The governing equations in terms of A_{mn} to E_{mn} are obtained by the principle of minimum total potential and are solved to obtain the coefficients and, hence, displacements, strains, etc.

Numerical Results

Different laminates – with symmetric, antisymmetric, and unsymmetricstacking – were analyzed using the present approach. However, detailed results will be presented only for the following cases.

Case 1: $(0/90\ \text{deg})_4$, 0 deg laminate under $q = q_0 \sin(\pi x/a) \sin(\pi y/b)$ (see Table 1).

Case 2: (0/90/0 deg) laminate subjected to 1) Uniformly Distributed Load UDL, 2) linearly varying load $q = q_0\ (x/a)$ and 3) central point load P (see Table 2).

The results for the other cases will be briefly discussed in the next section.

Table 1 Nondimensionalized deflection and stresses in nine-ply symmetric square laminates under sinusoidal loading

a/h	Source	$\bar{w}(a/2,b/2,0)$	$\bar{\sigma}_x(a/2,b/2,h/2)$	$\bar{\tau}_{xy}(0,0,h/2)$	$\bar{\tau}_{xz}(0,b/2,0)$	$\bar{\tau}_{yz}(a/2,0,0)$	$\bar{\sigma}_y(a/2,b/2,2h/5)$
4	Present	4.0507	0.7217	−0.03295	0.2106	0.2211	0.6132
	Pagano[7]	4.079	0.649, −0.684	−0.0328, +0.0337	0.223	0.223	0.628
10	Present	1.5079	0.5578	−0.0233	0.2449	0.2266	0.4754
	Pagano[7]	1.512	0.551	−0.0235, +0.0233	0.247	0.226	0.477
20	Present	1.1281	0.5421	−0.02180	0.2546	0.2215	0.4439
	Pagano[7]	1.129	0.541	−0.0219	0.255	0.221	0.444
100	Present	1.005	0.5388	−0.0213	0.2587	0.2188	0.4315
	Pagano[7]	1.005	0.539	−0.0213	0.259	0.219	0.431

$\bar{w} = 100E_T w/q_0 h s^4$, $(\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}) = (1/q_0 s^2)(\sigma_x, \sigma_y, \tau_{xy})$, $(\bar{\tau}_{xz}, \bar{\tau}_{yz}) = (1/q_0 s)(\tau_{xz}, \tau_{yz})$ where $s = (a/h)$.

Table 2 Nondimensionalized deflection and stresses for various loadings ($a/b = 1$, 0/90/0 deg

		Uniformly distributed load q_0					Central concentrated load P					Linearly varying load (q_0x/a)			
a/h	R[b]	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$	R[b]	w^*	σ_x^*	σ_y^*	τ_{xy}^*	R[c]	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\tau}_{xy}$
	9	3.0049				49	11.4073				9	1.0550			
		(2.9103)[a]													
	19	3.0041				199	11.4328				19	1.0547			
4	29	3.0041				229	11.4328				29	1.0547			
		(2.9091)													
	49	3.0041	1.2019	0.8311	−0.7486	239	11.4328	43.1872	40.3687	−1.4169	49	1.0547	0.4018	0.3269	0.0279
		(2.9091)													
	9	1.1576				49	4.4279				9	0.5383			
		(1.0903)													
	19	1.1573				199	4.4508				19	0.5381			
10	29	1.1573				229	4.4508				29	0.5381			
		(1.0900)													
	49	1.1573	0.8769	0.3615	−0.0419	239	4.4508	34.9279	31.1488	−1.1411	49	0.5381	0.3647	0.1767	0.0189
		(1.0900)													
	9	0.7961				49	2.8334				9	0.3799			
		(0.7761)													
	19	0.7960				199	2.8501				19	0.3799			
20	29	0.7960				299	2.8501				29	0.3799			
		(0.7760)													
	49	0.7960	0.8263	0.2380	−0.0333	239	2.8501	28.9505	24.5791	−0.9661	49	0.3799	0.3595	0.1192	0.0157
		(0.7760)													
	9	0.6714				49	2.1659				9	0.3253			
		(0.6705)													
	19	0.6713				199	2.1684				19	0.3253			
100	29	0.6713				229	2.1684				29	0.3253			
		(0.6705)													
	49	0.6713)	0.8084	0.1934	−0.0301	239	2.1684	15.6423	12.6314	−0.6174	49	0.3253	0.3578	0.0973	0.0144
		(0.6705)													

[a]Values inside the parentheses refer to Ref. 6.
[b]$m = n = 1, 3, 5, \ldots R$.
[c]$m = 1, 2, 3, \ldots R$ and $n = 1, 3, 5, \ldots R$.
$w^* = \dfrac{wbh^3}{Pa^3}$, $\sigma_x^* = \dfrac{\sigma_x bh^2}{Pa}$, $\sigma_y^* = \dfrac{\sigma_y bh^2}{Pa}$, and $\tau^* = \tau_{xy} bh^2$

The following engineering constants have been used: $E_L = 25 \times 10^6$ psi, $E_T = 10^6$ psi, $G_{LT} = 0.5 \times 10^6$ psi, $G_{TT} = 0.2 \times 10^6$ psi, $\nu_{LT} = \nu_{TT} = 0.25$.

In-plane stresses are computed using the constitutive equations, whereas the transverse shear stresses are obtained from the three-dimensional equilibrium equations. Problems under case 2 have been solved by the method of Fourier expansion.

Discussion

The present results are in good agreement with the elasticity solution[7] for the nine-ply symmetric laminates as shown in Table 1. Limited studies of other laminates indicate that the present model is more accurate than the corresponding smeared laminate theory[6] for all cases except that of the (90/0 deg) laminate and a very thick (0/90 deg)$_s$ laminate ($a/h = 4$). The present model is based on a symmetric shear stress distribution through the thickness. The corresponding displacement distribution is not realistic in (90/0 deg) laminate, and hence the poor performance cited above can be expected.

It can be seen from Table 2 that the series for the displacement variables converges faster than in the smeared laminate model for UDL (10×10 terms as against 20×20 terms of Ref. 6). For the linearly varying and central concentrated loads, the convergence is obtained with 10×10 and 100×100 terms, respectively. The authors believe that the results presented here for symmetric and asymmetric loadings would serve as reference for finite-element analyses.

Finally, it is of interest to see at what additional labor the improvement in accuracy over Ref. 6 is arrived at in the present theory. Additional computation is required only for the evaluation of r_k and t_k [Eq. (2)], which has to be done only once for all layers and involves simple arithmetic. Thus the cost of analysis is believed to be more or less equal to that of Ref. 6.

The present formulation can be extended to study the buckling and free vibration behavior of laminated plates.

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